# On the relationships between the fixed-$f_1$, fixed-$f_2$, and fixed-ratio phase derivatives of the $2f_1-f_2$ distortion product otoacoustic emission

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For primary frequency ratios, $f_2/f_1$, in the range 1.1–1.3, the fixed-$f_1$ (''$f_2$-sweep'') phase derivative of the $2f_1-f_2$ distortion product otoacoustic emission (DPOAE) is larger than the fixed-$f_2$ (''$f_1$-sweep'') one. It has been proposed by some researchers that part or all of the difference between these delays may be attributed to the so-called cochlear filter ''build-up'' or response time in the DPOAE generation region around the $f_2$ tonotopic site. The analysis of an approximate theoretical expression for the DPOAE signal [Talmadge *et al.*, J. Acoust. Soc. Am. **104**, 1517–1543 (1998)] shows that the contributions to the phase derivatives associated with the cochlear filter response is small. It is also shown that the difference between the phase derivatives can be qualitatively accounted for by assuming the approximate scale invariance of cochlear mechanics. The effects of DPOAE fine structure on the phase derivative are also explored, and it is found that the interpretation of the phase derivative in terms of the phase variation of a single DPOAE component can be quite problematic. 
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## I. INTRODUCTION

Distortion product otoacoustic emissions (DPOAEs) (Kemp, 1979) are pressure signals resulting from the cochlear nonlinear interaction of two external primary tones of frequency $f_1$ and $f_2$ ($f_2>f_1$) that may be measured in the ear canal. Studies of the correlation of DPOAE levels with hearing loss, and the effects of suppressor tones, noise exposure, and ototoxic drugs on DPOAEs, have suggested that these emissions may constitute clinically important indicators of the health status of the nonlinear active cochlear function (see, e.g., the reviews in Robinette and Glattke, 1997).

The distortion product (DP) energy is initially generated in the strong overlap region (or ''generation region'') of the primary tone cochlear activity patterns (around the $f_2$ tonotopic place). The generated DP waves travel in both the apical and basal directions. In the case of apical DPOAEs such as the one with the frequency $2f_1-f_2$, in which the DPOAE tonotopic site is apical to the $f_1$ and $f_2$ sites, the initially apical moving generated DP wave may be reflected by cochlear inhomogeneities around the DP tonotopic place (Shera and Zweig, 1993; Zweig and Shera, 1995). The resulting interference of this reflected wave (the ''reflection site component'') and the initially basally generated wave (the ''generator site component'') is observed as DPOAE fine structure in the ear canal, in which the DPOAE level varies with DPOAE frequency in a pseudoperiodic fashion (e.g., He and Schmiedt, 1993, 1996, 1997; Kummer *et al.*, 1995; Brown *et al.*, 1996; Gaskill and Brown, 1996). A comprehensive theoretical description of DPOAE fine structure as well as the fine structure of other types of otoacoustic emissions and the psychoacoustic microstructure of the hearing threshold has been recently developed (Talmadge *et al.*, 1996, 1997, 1998, 1999a). It should be noted that Kim (1980) was actually first to suggest cochlear wave reflection around the DP tonotopic site as the agent for DPOAE fine structure. A similar suggestion was made by Kemp and Brown (1983).

The phase and phase derivatives of DPOAEs as functions of the primary frequency have been of great interest to many researchers. As discussed in Talmadge *et al.* (1999a), the phase derivatives complement in many respects the usual DPOAE level measurements, and comparisons of the two sets of quantities provide additional information on the underlying mechanics responsible for DPOAE generation. Because of this potential for new theoretical insights, it is useful to have a quasirealistic analytic description of the DPOAE signal. The model of Talmadge *et al.* (1998) provides such a description in terms of the nonlinear component of the cochlear partition dynamics and the cochlear response to single-tone stimuli. Moreover, it can be used to give a connection between the temporal DPOAE response to pulsed tones (from which physical time delays may be inferred) and the steady-state response (Talmadge *et al.*, 1999b; Tubis *et al.*, 2000).

This paper will be focused mainly on the differences between $\tau_{\text{fixed-}f_1}$ and $\tau_{\text{fixed-}f_2}$, the negatives of the respective fixed-$f_1$ and -$f_2$ phase derivatives (which are usually referred to as group delays) of the $f_{dp}=2f_1-f_2$ DPOAE (e.g., O'Mahoney and Kemp, 1995; Bowman *et al.*, 1997, 1998; Schneider *et al.*, 1999a,b). Specifically, if the complex DPOAE signal is written as

$$S_{dp}=A_{dp}e^{i\varphi_{dp}}, \tag{1}$$

where $A_{dp}$ and $\varphi_{dp}$ are the real amplitude and phase of the $2f_1-f_2$ DPOAE, respectively, with the convention that the time dependence is $\exp(2\pi i f_{dp})$, then

$$\tau_{\text{fixed-}f_1}=-\frac{1}{2\pi}\left.\frac{\partial\varphi_{dp}}{\partial f_{dp}}\right|_{f_1}, \tag{2}$$

$$\tau_{\text{fixed-}f_2}=-\frac{1}{2\pi}\left.\frac{\partial\varphi_{dp}}{\partial f_{dp}}\right|_{f_2}. \tag{3}$$

Both of these phase derivatives are generally functions of $f_1$ and $f_2$. Bowman *et al.* (1997, 1998) have proposed that if $\tau_{\text{fixed-}f_1}$ and $\tau_{\text{fixed-}f_2}$ are interpreted as physical time delays, then all or a significant part of the difference between them is associated with the ''filter build-up'' time or response time of the cochlear elements around the $f_2$ tonotopic site. This proposal is evaluated in the present paper using the theoretical framework of the Talmadge *et al.* (1998) model. *The filter referred to by Bowman et al. (1997, 1998) will be assumed to be associated with the resonance response of the cochlear partition in the neighborhood of the* $f_2$ *tonotopic place (the initial DP generation region).*

It will be shown that the effect of this cochlear filter on the DPOAE phase and phase derivatives is very small, and that the approximate scale invariance of cochlear mechanics (e.g., Shera and Zweig, 1993; Zweig and Shera, 1995) can give rise to a substantial difference between $\tau_{\text{fixed-}f_1}$ and $\tau_{\text{fixed-}f_2}$ in the range of primary frequency ratios $f_2/f_1$ between 1.1 and 1.3, without the need for a significant contribution from the cochlear filter (Talmadge *et al.*, 1999b; Schneider *et al.*, 1999b; Tubis *et al.*, 2000; Prijs *et al.*, 2000). In particular, it is shown in the papers just cited that

$$\frac{\tau_{\text{fixed-}f_1}}{\tau_{\text{fixed-}f_2}}\approx\frac{2f_1}{f_2}, \tag{4}$$

which is in qualitative agreement with typical data (e.g., Bowman *et al.*, 1997; Schneider *et al.*, 1999a).

In order to simplify the discussion, only $f_{dp}=2f_1-f_2$ DPOAEs for which fine structure effects are negligible will be considered at first; i.e., reflection of apically moving DP waves from around the $f_{dp}$ tonotopic site will not be taken into account initially. [Time-domain and frequency-domain techniques for separating the two DPOAE components have been recently presented (Talmadge *et al.*, 1998, 1999a,b).] Also, $\tau_{\text{fixed-}f_1}$ and $\tau_{\text{fixed-}f_2}$, as defined above in terms of the steady-state DPOAE phase, are considered without discussing to what extent they correspond to physical time delays. In other papers (Talmadge *et al.*, 1999b; Tubis *et al.*, 2000), it is shown that considerable care must be exercised in attempts to associate them with physical time delays. It is also found that DPOAE fine structure can have a dramatic effect on $\tau_{\text{fixed-}f_1}$ and $\tau_{\text{fixed-}f_2}$, and that the interpretation of these delays in terms of the phase behavior of a DPOAE component originating from a single cochlear source can give very misleading results.

The essential elements of the Talmadge *et al.* (1998) model for the DPOAE are reviewed in Sec. II. The extent to which the cochlear filter behavior around the $f_2$ tonotopic site contributes to the DPOAE phase behavior is discussed in Sec. III using an analytic argument and several simplifying approximations. With the assumption that these approximations do not grossly distort the results, it is found that this contribution is fairly small and cannot be the main source of the difference between $\tau_{\text{fixed-}f_1}$ and $\tau_{\text{fixed-}f_2}$. This conclusion is supported by extensive numerical calculations in Sec. VII, in which the above-mentioned approximations are *not* made. In Sec. IV, the derivation of results such as Eq. (4) is reviewed using (1) simplified forms for the DPOAE phase and phase derivatives and the assumption that the generator region component is dominant, and (2) the assumption of scale invariance (e.g., Shera and Zweig, 1993; Zweig and Shera, 1995; Talmadge *et al.*, 1998). Group delays are obtained in Section V for the case in which only the reflection site component is important. Unlike the situation in Sec. IV, it is found that the phase derivatives are approximately independent of the measurement paradigm, and correspond to the approximate round-trip travel time of a signal at the DP frequency from the cochlear base to its tonotopic (reflection) site and back to the base. The case in which neither of the DPOAE components may be neglected is considered in Sec. VI. Here, it is found that significant deviations of the group delays from those for a dominant generator region (or reflection region) component can be present. The various assumptions used in obtaining the results in Secs. III, IV, V, and VI are justified in Sec. VII via numerical calculations. The fixed-$f_1$ and fixed-$f_2$ group delays for higher-order apical DPOAEs are considered in Sec. VIII, and a summary of results and conclusions is presented in Sec. IX. An outline of the perturbative method used to obtain the pressure wave equation for the $2\omega_1-\omega_2$ DP is given in the Appendix.

## II. MODEL FOR DPOAE PHASE

The analysis of this paper is based on an approximate expression for the DPOAE signal derived in Talmadge *et al.* (1998), to which the reader is referred for a detailed definition of symbols, and discussions of assumptions and approximations involved.

For a steady-state ''calibrated'' (complex) driving pressure $P_{dr}(t)$, given by

$$P_{dr}(t)=P_{dr}e^{i\omega t}, \tag{5}$$

the steady-state complex DPOAE amplitude of frequency $\omega_{dp}=2\omega_1-\omega_2$ is given by (Talmadge *et al.*, 1998)

$$P^{ss}_{dp}(\omega_1,\omega_2,\omega_{dp}) = \frac{P_l(\omega_1,\omega_2,\omega_{dp})+R_a(\omega_{dp})P_r(\omega_1,\omega_2,\omega_{dp})}{1-R_a(\omega_{dp})R_b(\omega_{dp})}, \quad (6)$$

where

$$P_l(\omega_1,\omega_2,\omega_{dp}) = -T_{pd}(\omega_{dp})\frac{k_{ow}(\omega_{dp})}{k_r(\omega_{dp})+k_{ow}(\omega_{dp})} \times \frac{\rho_p(\omega_1,\omega_2)}{\Delta_{sm}(0,\omega_{dp})}F_r(\infty,\omega_1,\omega_2), \quad (7)$$

$$P_r(\omega_1,\omega_2,\omega_{dp}) = -T_{pd}(\omega_{dp})\frac{k_{ow}(\omega_{dp})}{k_r(\omega_{dp})+k_{ow}(\omega_{dp})} \times \frac{\rho_p(\omega_1,\omega_2)}{\Delta_{sm}(0,\omega_{dp})}F_l(\infty,\omega_1,\omega_2). \quad (8)$$

[As usual, the complex temporal dependence is $\exp(i\omega_{dp}t)$.]

Here, $P_l(\omega_1,\omega_2,\omega_{dp})$ is the amplitude of the contribution to the DPOAE ear-canal pressure directly from the DP generation region, $R_a(\omega_{dp})P_r(\omega_1,\omega_2,\omega_{dp})$ is the amplitude from the reflection-site contribution, and $R_a(\omega_{dp})$ and $R_b(\omega_{dp})$ are, respectively, the cochlear apical and basal reflectances. As noted in Talmadge *et al.* (1999a), the quantity $P_r(\omega_1,\omega_2,\omega_{dp})$ gives the ear-canal contribution from the reflection-site component in the limit that $R_a(\omega_{dp})\rightarrow 1$ and $R_b(\omega_{dp})\rightarrow 0$. As noted in the Introduction, the reflection from the DP site will be neglected for the initial modeling considerations, so that Eq. (6) becomes

$$P^{ss}_{dp}(\omega_1,\omega_2,\omega_{dp}) \approx P_l(\omega_1,\omega_2,\omega_{dp}). \quad (9)$$

Some of the relevant expressions for the terms in Eqs. (7) and (8) are

$$F_{r,l}(x,\omega_1,\omega_2) = \int_0^x dx'\ \gamma_0(x')\chi_r^2(x',\omega_1) \times \chi_r^*(x',\omega_2)\chi_{r,l}(x',\omega_{dp}), \quad (10)$$

$$\chi_r(x,\omega) = \frac{\Delta_{sm}(0,\omega)}{\Delta_{sm}(x,\omega)}\psi_r(x,\omega), \quad (11)$$

$$\psi_r(x,\omega) \cong \sqrt{\frac{k(0,\omega)}{k(x,\omega)}}\exp\left\{-i\int_0^x k(x',\omega)dx'\right\}, \quad (12)$$

$$\psi_l(x,\omega) \cong \sqrt{\frac{k(0,\omega)}{k(x,\omega)}}\exp\left\{+i\int_0^x k(x',\omega)dx'\right\}, \quad (13)$$

$$k(x,\omega) = \frac{k_0\omega}{\sqrt{\Delta_{sm}(x,\omega)}}, \quad (14)$$

$$\Delta_{sm}(x,\omega) = \omega_0^2(x)-\omega^2+i\omega\gamma_0(x) + \rho f\omega_0^2(x)e^{-i\psi_f\omega/\omega_0(x)} + \rho_s\omega_0^2(x)e^{-i\psi_s\omega/\omega_0(x)}, \quad (15)$$

$$\rho_p(\omega_1,\omega_2) = \frac{i\sigma_{bm}k_0^2(2\omega_1-\omega_2)^3 b_r^2(0,\omega_1)b_r^*(0,\omega_2)}{b_{nl}^2}, \quad (16)$$

$$b_r(0,\omega) = \frac{-G_{me}(\omega)k_{ow}(\omega)P_{dr}(\omega)}{\sigma_{bm}\Delta_{sm}(0,\omega)(k_r(0,\omega)+k_{ow}(\omega))}, \quad (17)$$

$$\omega_0(x) \cong \omega_{0c}e^{-k_\omega x}, \quad (18)$$

where $b_{nl}$ is the nonlinear saturation level. The reader is referred to Talmadge *et al.* (1998) for the explanation of the above formulas, as well as further discussion of the notation. It should be noted that Talmadge *et al.* (1998) further assumed that

$$\gamma_0(x) = \varepsilon_\gamma\omega_0(x), \quad (19)$$

where $\varepsilon_\gamma = \gamma_0(x)/\omega_0(x)$ is a constant in their analysis. In terms of their notation,

$$F_{r,l}(x,\omega_1,\omega_2) \cong \varepsilon_\gamma I_{r,l}(x,\omega_1,\omega_2), \quad (20)$$

$$\varepsilon_\gamma\rho_p(\omega_1,\omega_2) = \rho_0(\omega_1,\omega_2), \quad (21)$$

$$I_{r,l}(x,\omega_1,\omega_2) = \int_0^x dx'\ \omega_0(x')\chi_r^2(x',\omega_1) \times \chi_r^*(x',\omega_2)\chi_{r,l}(x',\omega_{dp}). \quad (22)$$

Equation (19) will be assumed in the derivation of the results. This assumption will be relaxed in Sec. VII, where the semianalytic results are compared to those obtained from a numerical model in which $\varepsilon_\gamma \neq$ constant. The WKB approximation (e.g., Zweig *et al.*, 1976) is used for the cochlear traveling wave basis function $\psi_r(x,\omega)$, along with the Zweig (1991) form of the cochlear macromechanical transpartition impedance. In order to illustrate in part how these equations were obtained, and to clarify the discussion of the numerical results in Sec. VII, the perturbative analysis for the $2\omega_1-\omega_2$ DP is outlined in the Appendix.

## III. COCHLEAR FILTER CONTRIBUTION TO THE DPOAE PHASE

In this section, an analytic argument, based on a number of simplifying approximations (some of which may appear dubious to the reader), is used to make plausible the smallness of the cochlear filter contribution to the DPOAE phase. In Sec. VII, numerical calculations, in which these approximations are *not* made, are shown to confirm the results found here. Therefore, a reader who is not interested in analytic details need only read through this section lightly in order to see where in the formal structure of the DPOAE pressure signal the cochlear filter contribution to the phase arises.

Within the approximations of Eq. (9), the DPOAE phase is that of $I_r(\infty,\omega_1,\omega_2)$ [see Eqs. (8), (10), (19), and (20)] except for a small contribution from the acousto-mechanics of the ear canal and middle ear. Consequently, the focus will be on this factor in the remainder of this section.

Most of the contribution to the integral $I_r(\infty,\omega_1,\omega_2)$ comes from the region around the amplitude maximum of $\chi_r^*(x,\omega_2)$ at

$$x \approx \hat{x}(\omega_2) = \frac{1}{k_\omega}\log\left(\frac{\omega_{0c}}{\omega_2}\right). \quad (23)$$

In order to estimate the phase of $I_r(\infty,\omega_1,\omega_2)$, it is convenient to expand the $x$ dependence of the phase of the integrand of $I_r(\infty,\omega_1,\omega_2)$ in Eq. (22) about $x=\hat{x}(\omega_2)$ to first order in $x-\hat{x}(\omega_2)$. If the phase contributions from the $\Delta_{sm}(x,\omega)$ factors are not taken into account, this phase dependence is approximately given by

$$\begin{aligned}&\arg[\text{integrand of } I_r(\infty,\omega_1,\omega_2)]\\&\quad\cong\int_0^{\hat{x}(\omega_2)}dx'\,\mathrm{Re}[-2k(x',\omega_1)+k^*(x',\omega_2)\\&\quad\quad-k(x',\omega_{dp})]+\mathrm{Re}[-2k(\hat{x}(\omega_2),\omega_1)\\&\quad\quad+k^*(\hat{x}(\omega_2)),\omega_2)-k(\hat{x}(\omega_2),\omega_{dp})][x-\hat{x}(\omega_2)],\end{aligned}\tag{24}$$

where $\mathrm{Re}\,z$ denotes the real part of $z$, and $\arg z$ denotes the phase of $z$.

It will be shown in Sec. IV that, at least for the case of a scale-invariant wave number function $k(x,\omega)$, the first term on the right-hand side of Eq. (24) does not depend significantly on the behavior of $k(x,\omega_2)$ in the resonance region, $x\approx\hat{x}(\omega_2)$ [see, e.g., Eq. (46)]. Thus, the importance of the sharpness of the filter function for the $\omega_2$ primary in determining the DPOAE phase may be assessed by estimating the difference between the phase of $I_r(\infty,\omega_1,\omega_2)$ and this first term. This difference was previously estimated by Talmadge *et al.* (1998) and shown to be small (see their Secs. V B and C). However, they neglected the additional phase of the integrand of $I_r(\infty,\omega_1,\omega_2)$ which comes from the factor $\Delta^*_{sm}(x,\omega_2)$. For present purposes, therefore, it is sufficient to calculate the phase difference with this additional phase dependence included in the integrand in order to demonstrate the relative importance of the sharpness of the filter function (and thus the relative importance of the filter build-up time). A plausibility argument will now be given to demonstrate that the phase difference remains small even if the additional phase is included in the integrand, and thus that the character of the resonance region for the $\omega_2$ primary is not significant in determining the DPOAE phase.

In order to simplify the evaluation of this extra phase contribution, the time-delayed stiffness terms in $[\Delta^*_{sm}(x,\omega_2)]^{-3/4}$ in Eq. (15) are dropped. These terms are however still included in the numerical treatment of this problem in Sec. VII in order to assess their relative importance. In addition, $\gamma_0$ is assumed to be constant, the magnitude of $\chi^*_r(x,\omega_2)$ around $x\approx\hat{x}(\omega_2)$ is approximated by (e.g., Zweig and Shera, 1995; Talmadge *et al.*, 1998)

$$|\chi^*_r(x,\omega_2)|\cong|\hat{\chi}_r(\omega_2)|e^{-[x-\hat{x}(\omega_2)]^2/2\sigma_x^2},\tag{25}$$

$|\chi_r(x,\omega_1)|$ is replaced by its value at $x=\hat{x}(\omega_2)$, $\omega_0(x)$ is evaluated at $\hat{x}(\omega_2)$, and $[\Delta^*_{sm}(x,\omega_2)]^{3/4}$ is expanded about $x\approx\hat{x}(\omega_2)$

$$\begin{aligned}[\Delta^*_{sm}(x,\omega_2)]^{3/4}&\approx(-i\omega_2\gamma_0)^{3/4}+\frac{3}{4}\frac{1}{(-i\omega_2\gamma_0)^{1/4}}\\&\quad\times(-2k_\omega\omega_2^2)[x-\hat{x}(\omega_2)]\\&=-\frac{3}{2}k_\omega\omega_2^2\frac{1}{(-i\omega_2\gamma_0)^{1/4}}\\&\quad\times\left[x-\hat{x}(\omega_2)+\frac{2i\gamma_0}{3k_\omega\omega_2}\right].\end{aligned}\tag{26}$$

Since the phase of

$$\frac{-3}{2}k_\omega\omega_2^2\frac{1}{(-i\omega_2\gamma_0)^{1/4}}\tag{27}$$

is constant (i.e., independent of $\omega_1$ and $\omega_2$), the $\omega_1$- and $\omega_2$-dependent part of this phase contribution can be approximately represented as

$$\begin{aligned}&\arg\int_0^\infty dx'\exp\{i\,\mathrm{Re}[-2k(\hat{x}(\omega_2),\omega_1)+k^*(\hat{x}(\omega_2),\omega_2)\\&\quad-k(\hat{x}(\omega_2),\omega_{dp})][x'-\hat{x}(\omega_2)]\}\\&\quad\times\frac{|x'-\hat{x}(\omega_2)+iL|}{x'-\hat{x}(\omega_2)+iL}\exp\left\{-\frac{[x-\hat{x}(\omega_2)]^2}{2\sigma_x^2}\right\},\end{aligned}\tag{28}$$

where $L\approx2\gamma_0/3k_\omega\omega_2\sim0.1\,\mathrm{cm}$. Using the variable substitutions

$$x''=x'-\hat{x}(\omega_2),\tag{29}$$

$$\begin{aligned}K=&\mathrm{Re}[-2k(\hat{x}(\omega_2),\omega_1)+k^*(\hat{x}(\omega_2),\omega_2)\\&-k(\hat{x}(\omega_2),\omega_{dp})],\end{aligned}\tag{30}$$

the phase contribution given by Eq. (28) may be written approximately as

$$\begin{aligned}&\arg\int_{-\infty}^\infty dx''\,e^{-(x'')^2/2\sigma_x^2}e^{iKx''}\frac{x''-iL}{\sqrt{(x'')^2+L^2}}\\&\quad=\arg i\int_{-\infty}^\infty dx''\,e^{-(x'')^2/2\sigma_x^2}\frac{x''\sin Kx''-L\cos Kx''}{\sqrt{(x'')^2+L^2}}\\&\quad=\pm\frac{\pi}{2}.\end{aligned}\tag{31}$$

Since Eq. (31) yields a constant phase, the frequency derivatives of this extra phase contribution are vanishingly small within the approximation scheme used here and are probably very small in general. The $\omega_1$- and $\omega_2$-dependent part of the phase is approximately given by

$$\begin{aligned}\varphi_{\mathrm{gen}}(\omega_1,\omega_2,\omega_{dp})&\equiv\arg[P^{ss}_{dp}(\omega_1,\omega_2,\omega_{dp})]\\&\cong\arg[I_r(\infty,\omega_1,\omega_2)]\\&\cong\int_0^{\hat{x}(\omega_2)}dx'\,\mathrm{Re}[k^*(x',\omega_2)\\&\quad-2k(x',\omega_1)-k(x',\omega_{dp})],\end{aligned}\tag{32}$$

where the subscript gen indicates that this is the contribution to the DPOAE phase only from the DP generation region. Note that this equation simply states that the DPOAE phase is given by $2\pi\times$[the number of $\omega_2$ wavelengths—twice the number of $\omega_1$ wavelengths—the number of $\omega_{dp}$ wavelengths of the associated traveling waves between the cochlear base and the $\omega_2$ tonotopic place].

The calculations of this section are admittedly based on many approximations, including the use of a symmetric [Eq. (25)] rather than a (more realistic) highly asymmetric cochlear activity pattern. However, it will be shown in Sec. VII that numerical calculations, *in which none of the approxima-*

*tions used in this section is made*, support the general conclusion that taking into account the phase of $\Delta_{sm}^*(x,\omega_2)$, and hence the most important effects of the cochlear filter on the DPOAE phase, gives only very small contributions to the DPOAE phase derivatives.

## IV. APPROXIMATE PHASE DERIVATIVES FROM SCALE-INVARIANCE

In this section, it will be demonstrated that the substantial difference between the fixed-$f_1$ and fixed-$f_2$ phase derivatives can be understood as a consequence of the approximate scale invariance of the cochlear mechanics (e.g., Shera and Zweig, 1993; Zweig and Shera, 1995; Talmadge *et al.*, 1998) applied to the evaluation of $\varphi_{\text{gen}}(\omega_1,\omega_2,\omega_{dp})$ given by Eq. (32).

Scale invariance of the form

$$k(x,\omega)\cong k\left(\frac{\omega}{\omega_0(x)}\right), \tag{33}$$

will be assumed for the wave number function, where $\omega_0(x)$ is given by Eq. (18). Note that the assumption Eq. (19) gives a scale-invariant wave number. From Eqs. (18) and (33), it follows that

$$\frac{\partial k(x,\omega)}{\partial \omega}\cong\frac{\partial k(\omega/\omega_0(x))}{\partial \omega}\cong\frac{1}{\omega_0(x)}k'(\omega/\omega_0(x)), \tag{34}$$

$$\frac{\partial k(x,\omega)}{\partial x}\cong\frac{\partial k(\omega/\omega_0(x))}{\partial x}\cong\frac{k_\omega\omega}{\omega_0(x)}k'(\omega/\omega_0(x)). \tag{35}$$

[The primes indicate differentiation of $k$ with respect to its argument, $\omega/\omega_0(x)$.] Comparison of Eqs. (34) and (35) gives the approximate relationship

$$\frac{\partial k(x,\omega)}{\partial \omega}\cong\frac{1}{k_\omega\omega}\frac{\partial k(x,\omega)}{\partial x}. \tag{36}$$

The phase derivative,

$$\tau_{\text{gen}}(\omega_1,\omega_2,\omega_{dp})\equiv-\frac{\partial\varphi_{\text{gen}}(\omega_1,\omega_2,\omega_{dp})}{\partial\omega_{dp}}, \tag{37}$$

is usually evaluated under one of three conditions: (i) $\omega_1=\text{constant}$ [fixed-$f_1$ or $f_2$-sweep paradigm]; (ii) $\omega_2=\text{constant}$ [fixed-$f_2$ or $f_1$-sweep paradigm]; (iii) $\omega_2/\omega_1=\text{constant}$ [fixed-$f_2/f_1$ or fixed-ratio paradigm]. Since $\omega_{dp}=2\omega_1-\omega_2$, the derivatives of $\omega_1$ and $\omega_2$ with respect to $\omega_{dp}$ can be expressed in terms of the constants

$$\alpha_1=\frac{\partial\omega_1}{\partial\omega_{dp}},\quad \alpha_2=\frac{\partial\omega_2}{\partial\omega_{dp}}, \tag{38}$$

where

$$\alpha_1=0,\quad \alpha_2=-1:\quad (\omega_1=\text{constant}), \tag{39}$$

$$\alpha_1=\tfrac{1}{2},\quad \alpha_2=0:\quad (\omega_2=\text{constant}), \tag{40}$$

$$\alpha_1=\frac{1}{2-r},\quad \alpha_2=\frac{r}{2-r}:\quad (\omega_2/\omega_1=r=\text{constant}). \tag{41}$$

Also,

$$\frac{\partial\hat{x}(\omega_2)}{\partial\omega_{dp}}=\frac{\partial\hat{x}(\omega_2)}{\partial\omega_2}\frac{\partial\omega_2}{\partial\omega_{dp}}=-\frac{\alpha_2}{k_\omega\omega_2}. \tag{42}$$

Equations (32), (33), (37), and (42) give

$$\begin{aligned}\tau_{\text{gen}}(\omega_1,\omega_2,\omega_{dp})=&-\int_0^{\hat{x}(\omega_2)}dx'\ \text{Re}\left[\frac{\partial k^*(x',\omega_2)}{\partial\omega_{dp}}\right.\\&\left.-2\frac{\partial k(x',\omega_1)}{\partial\omega_{dp}}-\frac{\partial k(x',\omega_{dp})}{\partial\omega_{dp}}\right]\\&+\frac{\alpha_2}{k_\omega\omega_2}\text{Re}[k^*(\hat{x}(\omega_2),\omega_2)\\&-2k(\hat{x}(\omega_2),\omega_1)-k(\hat{x}(\omega_2),\omega_{dp})].\end{aligned} \tag{43}$$

Applying Eqs. (38) through (43) yields

$$\begin{aligned}\tau_{\text{gen}}(\omega_1,\omega_2,\omega_{dp})=&\frac{\alpha_2}{k_\omega\omega_2}\text{Re}[k^*(\hat{x}(\omega_2),\omega_2)\\&-2k(\hat{x}(\omega_2),\omega_1)-k(\hat{x}(\omega_2),\omega_{dp})]\\&-\int_0^{\hat{x}(\omega_2)}dx'\ \text{Re}\left[\alpha_2\frac{\partial k^*(x',\omega_2)}{\partial\omega_2}\right.\\&\left.-2\alpha_1\frac{\partial k(x',\omega_1)}{\partial\omega_1}-\frac{\partial k(x',\omega_{dp})}{\partial\omega_{dp}}\right],\end{aligned} \tag{44}$$

which can be further reduced using Eq. (36) to give

$$\begin{aligned}\tau_{\text{gen}}(\omega_1,\omega_2,\omega_{dp})=&\frac{\alpha_2}{k_\omega\omega_2}\text{Re}[k^*(\hat{x}(\omega_2),\omega_2)\\&-2k(\hat{x}(\omega_2),\omega_1)-k(\hat{x}(\omega_2),\omega_{dp})]\\&-\int_0^{\hat{x}(\omega_2)}dx'\ \text{Re}\left[\frac{\alpha_2}{k_\omega\omega_2}\frac{\partial k^*(x',\omega_2)}{\partial x'}\right.\\&-\frac{2\alpha_1}{k_\omega\omega_1}\frac{\partial k(x',\omega_1)}{\partial x'}\\&\left.-\frac{1}{k_\omega\omega_{dp}}\frac{\partial k(x',\omega_{dp})}{\partial x'}\right].\end{aligned} \tag{45}$$

After the $x'$ integration is carried out, this expression reduces to the final form

$$\begin{aligned}\tau_{\text{gen}}(\omega_1,\omega_2,\omega_{dp})=&\frac{2\ \text{Re}[k^*(\hat{x}(\omega_2),\omega_1)]}{k_\omega\omega_1}\left[\alpha_1-\alpha_2\frac{\omega_1}{\omega_2}\right]\\&+\frac{\text{Re}[k(\hat{x}(\omega_2),\omega_{dp})]}{k_\omega\omega_{dp}}\left[1-\alpha_2\frac{\omega_{dp}}{\omega_2}\right]\\&+\left\{\frac{\alpha_2}{k_\omega\omega_2}\text{Re}[k^*(0,\omega_2)]\right.\\&-\frac{2\alpha_1}{k_\omega\omega_1}\text{Re}[k(0,\omega_1)]\\&\left.-\frac{1}{k_\omega\omega_{dp}}\text{Re}[k(0,\omega_{dp})]\right\}.\end{aligned} \tag{46}$$

The wave number function $k(x,\omega)$ has the approximate scale-invariant form (Talmadge *et al.*, 1998)

$$k(x,\omega)\cong\frac{k_0\omega}{\sqrt{\omega_0^2(x)-\omega^2}}[=\text{real}],\quad \omega\ll\omega_0(x), \tag{47}$$

where $k_0$ is a constant (see Talmadge *et al.*, 1998). Evaluating Eq. (47) at $x=0$ for $\omega\ll\omega_{0c}$ gives

$$k(0,\omega)\cong\frac{k_0\omega}{\omega_{0c}}. \tag{48}$$

Since $\omega_{dp}=2\omega_1-\omega_2$, the $\alpha_{1,2}$ are then related by

$$2\alpha_1-\alpha_2=1, \tag{49}$$

which is true for any DPOAE measurement paradigm for which $\omega_{dp}$ is varied. Applying Eqs. (47), (48), and (49) to Eq. (46) gives

$$\tau_{\text{gen}}(\omega_1,\omega_2,\omega_{dp})=\left[\alpha_1-(2\alpha_1-1)\frac{\omega_1}{\omega_2}\right]\left(\frac{2k_0}{k_\omega}\right)\times\left(\frac{1}{\sqrt{\omega_2^2-\omega_1^2}}+\frac{1}{\sqrt{\omega_2^2-\omega_{dp}^2}}\right)-\frac{2k_0}{k_\omega\omega_{0c}}. \tag{50}$$

Using the values of $k_0$, $k_\omega$, and $\omega_{0c}$ for humans as quoted in Talmadge *et al.* (1998), $2k_0/k_\omega\omega_{0c}\cong 0.25\,\text{ms}$, which, for $\omega_{dp}$, $\omega_1$, $\omega_2\ll\omega_{0c}$, is very small compared to other phase derivative contributions associated with scale-invariance violations of the cochlear parameters, as well as those associated with the middle and outer ear. Consequently, the $2k_0/k_\omega\omega_{0c}$ term in Eq. (50) will be neglected for the purpose of this discussion. A brief description of the consequences of including this term is given at the end of this section.

From Eqs. (41) and (50), the fixed-ratio phase derivative is

$$\tau_{\text{fixed-}r}(\omega_{dp})\equiv\tau_{dp}(\omega_1,\omega_2,\omega_{dp})|_{\omega_2/\omega_1=r}\cong 0. \tag{51}$$

The fact that $\tau_{\text{fixed-}r}\cong 0$ for a cochlea which is approximately scale invariant was first pointed out by Kemp and Brown (1983). From Eqs. (39) and (50), the fixed-$f_1$ ($f_2$-sweep) phase derivative is

$$\tau_{\text{fixed-}f_1}=\tau_{\text{gen}}(\omega_1,\omega_2,\omega_{dp})|_{\omega_1}=\frac{2\omega_1}{\omega_2}\frac{k_0}{k_\omega}\left(\frac{1}{\sqrt{\omega_2^2-\omega_1^2}}+\frac{1}{\sqrt{\omega_2^2-\omega_{dp}^2}}\right). \tag{52}$$

Finally, using Eqs. (40) and (50), the fixed-$f_2$ ($f_1$-sweep) phase derivative is

$$\tau_{\text{fixed-}f_2}=\tau_{\text{gen}}(\omega_1,\omega_2,\omega_{dp})|_{\omega_2}=\frac{k_0}{k_\omega}\left(\frac{1}{\sqrt{\omega_2^2-\omega_1^2}}+\frac{1}{\sqrt{\omega_2^2-\omega_{dp}^2}}\right). \tag{53}$$

Thus, the ratio

$$\frac{\tau_{\text{fixed-}f_1}}{\tau_{\text{fixed-}f_2}}=\frac{2\omega_1}{\omega_2} \tag{54}$$

is significantly larger than 1 for $\omega_2/\omega_1$ in the range 1.1–1.3, as was observed experimentally by Bowman *et al.* (1997, 1998) and by Schneider *et al.* (1999a,b). Equation (54) may also be derived from the assumption of scale invariance of the phase (Schneider *et al.*, 1999b; Prijs *et al.*, 2000, and Sec. VIII of this paper). From the additional assumptions needed besides that of wave number scale invariance given by Eq. (33), it is evident that the assumption of phase scale invariance is far more restrictive than wave number scale invariance.

Equation (54) suggests the definition of a new quantity $\mathcal{S}(\omega_{dp})$ that indicates the degree of conformity of the DPOAE phase with the assumption of DPOAE phase scale invariance

$$\mathcal{S}(\omega_{dp})\equiv\left[\left(\frac{2\omega_1}{\omega_2}\right)^{-1}\frac{\tau_{\text{fixed-}f_1}(\omega_1,\omega_2,\omega_{dp})}{\tau_{\text{fixed-}f_2}(\omega_1,\omega_2,\omega_{dp})}\right]. \tag{55}$$

The factor $\mathcal{S}(\omega_{dp})$ is equal to 1 only if phase scale invariance within the cochlea holds, and is close to 1 only if approximate wave number scale invariance within the cochlea is assumed and, in addition, if the effects from the base and apex and from the middle and outer ears on the phase can be neglected. If middle-ear effects are included in the model, significant deviations of $\mathcal{S}$ from 1 should be expected around the resonance frequency of the middle ear. The effect of the neglected factor of $2k_0/k_\omega\omega_{0c}$ (which parametrizes the effects of the cochlear base) becomes relatively more important as the DP frequency is increased, so that a significant deviation of $\mathcal{S}$ from 1 is also expected at DP frequencies approaching $\omega_{0c}$.

It is interesting to compute the deviation of $\mathcal{S}$ from 1 if the assumption that the effects of the base are negligible is relaxed. This assumption amounts to the neglect of the term $2k_0/k_\omega\omega_{0c}$ in Eq. (50). With the assumption that this term is small but non-negligible, the factor $\mathcal{S}$ becomes

$$\mathcal{S}(\omega_{dp},r)\cong 1+(\omega_{dp}/\omega_{0c})f_{\mathcal{S}}(r), \tag{56}$$

$$f_{\mathcal{S}}(r)=\left(\frac{1}{\sqrt{r^2-1}}+\frac{1}{2\sqrt{r-1}}\right)^{-1}. \tag{57}$$

## V. APPROXIMATE PHASE DERIVATIVES OF APICAL DPOAE COMPONENT

In Secs. III and IV, the DPOAE phase derivatives were obtained under the assumption that the effects of reflection of the initially generated apical wave from around the DP tonotopic site could be neglected. In this section, the DPOAE phase derivatives will be derived assuming instead that $R_aP_r\gg P_l$, but that $|R_a||R_b|\ll 1$: that is, the DPOAE reflection-site component will be assumed dominant, but the effects of multiple internal reflection will be neglected. In this case, the DPOAE signal is

$$P_{dp}^{ss}(\omega_1,\omega_2,\omega_{dp})\cong R_a(\omega_{dp})P_r(\omega_1,\omega_2,\omega_{dp}). \tag{58}$$

Following the procedures outlined in the previous section, but now assuming that the dominant phase contribution to $P_r$ comes from $\arg I_l$, gives

$$\begin{aligned}\varphi_{\text{refl}}(\omega_1,\omega_2,\omega_{dp}) &\cong \arg[R_a(\omega_{dp})P_r(\omega_1,\omega_2,\omega_{dp})]\\ &\cong \arg[I_l(\infty,\omega_1,\omega_2,\omega_{dp})]+\arg[R_a(\omega_{dp})]\\ &\cong \int_0^{\hat{x}(\omega_2)} dx' \,\text{Re}[k^*(x',\omega_2)-2k(x',\omega_1)\\ &\quad +k(x',\omega_{dp})]+\varphi_a(\omega_{dp}),\end{aligned} \tag{59}$$

where $\varphi_a(\omega_{dp})\equiv \arg R_a(\omega_{dp})$, and where the subscript ''refl'' indicates that this phase represents the contribution only from the DPOAE reflection site component. From Zweig and Shera (1995) and Talmadge *et al.* (1998)

$$\varphi_a(\omega)\cong -\frac{2\hat{k}}{k_\omega}\log\left(\frac{\omega}{\omega_{0c}}\right), \tag{60}$$

where $\hat{k}=\hat{k}(\omega)\equiv \text{Re}[k(\hat{x}(\omega),\omega)]$ (assumed constant with respect to $\omega$ under scale invariance) is the real part of the wave number at the basilar membrane activity pattern maximum for an initially apical moving wave of frequency $\omega$. The phase derivative

$$\tau_{\text{refl}}(\omega_1,\omega_2,\omega_{dp})=-\frac{\partial\varphi_{refl}(\omega_1,\omega_2,\omega_{dp})}{\partial\omega_{dp}} \tag{61}$$

is given by

$$\begin{aligned}\tau_{\text{refl}}(\omega_1,\omega_2,\omega_{dp}) &= \frac{2\,\text{Re}[k(\hat{x}(\omega_2),\omega_1)]}{k_\omega\omega_1}\left[\alpha_1-\alpha_2\frac{\omega_1}{\omega_2}\right]\\ &\quad -\frac{\text{Re}[k(\hat{x}(\omega_2),\omega_{dp})]}{k_\omega\omega_{dp}}\left[1-\alpha_2\frac{\omega_{dp}}{\omega_2}\right]\\ &\quad +\left\{\frac{\alpha_2}{k_\omega\omega_2}\text{Re}[k(0,\omega_2)]\right.\\ &\quad -\frac{2\alpha_1}{k_\omega\omega_1}\text{Re}[k(0,\omega_1)]\\ &\quad \left.+\frac{1}{k_\omega\omega_{dp}}\text{Re}[k(0,\omega_{dp})]\right\}+\frac{2\hat{k}}{k_\omega\omega_{dp}}.\end{aligned} \tag{62}$$

Applying Eqs. (47), (48), and (49) to Eq. (62) gives

$$\begin{aligned}\tau_{\text{refl}}(\omega_1,\omega_2,\omega_{dp}) &= \frac{2\hat{k}}{k_\omega\omega_{dp}}-\left[\alpha_1-(2\alpha_1-1)\frac{\omega_1}{\omega_2}\right]\left(\frac{2k_0}{k_\omega}\right)\\ &\quad\times\left(\frac{1}{\sqrt{\omega_2^2-\omega_{dp}^2}}-\frac{1}{\sqrt{\omega_2^2-\omega_1^2}}\right).\end{aligned} \tag{63}$$

The interpretation of Eq. (63) may be facilitated by noting that the physical round-trip travel time of a wave packet of center frequency $\omega$ from the base to its tonotopic position and back to the base is just (Lighthill, 1978)

$$\begin{aligned}\tau_{rt}(\omega) &= 2\int_0^{\hat{x}(\omega)} dx'\frac{\partial k(x',\omega)}{\partial\omega}\\ &\cong \frac{2}{k_\omega\omega}\int_0^{\hat{x}(\omega)} dx'\frac{\partial k(x',\omega)}{\partial x'}\\ &= \frac{2}{k_\omega\omega}[\hat{k}(\omega)-k(0,\omega)]\approx\frac{2\hat{k}}{k_\omega\omega},\end{aligned} \tag{64}$$

where Eq. (36) has been used in obtaining this result. Comparing Eqs. (63) and (64), it is easy to see that the phase derivative for the DPOAE reflection-site component is just the (physical) round-trip travel time minus a small correction term. This correction term just accounts for the phase effects of the variation in the separation between the generation region and the DP reflection site. For fixed-$f_1$ and fixed-$f_2$ sweeps in the range of $r$ between 1.1 and 1.3, these corrections are less than 1/6 of $\tau_{\text{fixed-}f_1}$ and $\tau_{\text{fixed-}f_2}$, respectively, for the cases in which the generator region component is dominant [Eqs. (52) and (53)]. This is due to the minus sign between the two terms in the last factor in Eq. (63).

## VI. DPOAE PHASE-DERIVATIVE FINE STRUCTURE

In this section, the phase derivative for the case in which neither $P_l$ nor $R_aP_r$ is negligible but $|R_a||R_b|\ll 1$ will be considered. In this case, we have

$$\begin{aligned}P_{dp}^{ss}(\omega_1,\omega_2,\omega_{dp}) &\cong P_l(\omega_1,\omega_2,\omega_{dp})\\ &\quad +R_a(\omega_{dp})P_r(\omega_1,\omega_2,\omega_{dp}).\end{aligned} \tag{65}$$

To simplify the notation, the following definitions are introduced:

$$P_l(\omega_1,\omega_2,\omega_{dp})=A_{\text{gen}}(\omega_1,\omega_2,\omega_{dp})e^{i\varphi_{\text{gen}}(\omega_1,\omega_2,\omega_{dp})}, \tag{66}$$

$$\begin{aligned}R_a(\omega_{\text{refl}})P_r(\omega_1,\omega_2,\omega_{dp}) &= A_{\text{refl}}(\omega_1,\omega_2,\omega_{dp})\\ &\quad\times e^{i\varphi_{\text{refl}}(\omega_1,\omega_2,\omega_{dp})}.\end{aligned} \tag{67}$$

From the formalism of Talmadge *et al.* (1998), the amplitudes $A_{\text{gen}}$ and $A_{\text{refl}}$ are expected to vary slowly compared to $\varphi_{\text{refl}}$. Consequently, $A_{\text{gen}}$ and $A_{\text{refl}}$ will be treated as constants with respect to their arguments in the following analysis.

Using Eqs. (66) and (67), the phase of $P_{ss}^{dp}(\omega_1,\omega_2,\omega_{dp})$ is then given by

$$\begin{aligned}\varphi_{dp}(\omega_1,\omega_2,\omega_{dp}) &= \varphi_{\text{gen}}(\omega_1,\omega_2,\omega_{dp})\\ &\quad +\tan^{-1}\left[\frac{\mathcal{R}\sin(\Delta\varphi(\omega_1,\omega_2,\omega_{dp}))}{1+\mathcal{R}\cos(\Delta\varphi(\omega_1,\omega_2,\omega_{dp}))}\right],\end{aligned} \tag{68}$$

where $\mathcal{R}\equiv A_{\text{refl}}/A_{\text{gen}}$ and

$$\begin{aligned}\Delta\varphi(\omega_1,\omega_2,\omega_{dp}) &\equiv \varphi_{\text{refl}}(\omega_1,\omega_2,\omega_{dp})-\varphi_{\text{gen}}(\omega_1,\omega_2,\omega_{dp})\\ &= \varphi_{dp0}(\omega_{dp},\omega_2)+\varphi_a(\omega_{dp}),\end{aligned} \tag{69}$$

$$\varphi_{dp0}(\omega_{dp},\omega_2)=2\int_0^{\hat{x}(\omega_2)} dx'\,\text{Re}[k(x',\omega_{dp})], \tag{70}$$

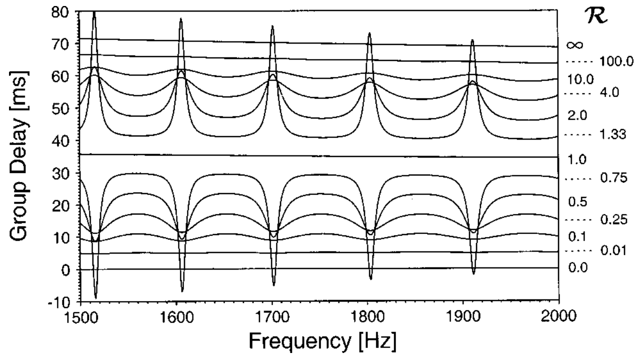


FIG. 1. Plot of the fixed-ratio paradigm phase derivative $\tau_{dp}$ using Eq. (74), for various admixtures $\mathcal{R}$ of the generation region and reflection-site DPOAE components. In order to be distinguishable, each successive curve is shifted vertically on the scale by 5 ms.

where Eq. (70) follows from Eqs. (32) and (59). The phase derivative for this case is then

$$\tau_{dp}(\omega_1,\omega_2,\omega_{dp})=\left[\frac{1+\mathcal{R}\cos\Delta\varphi(\omega_1,\omega_2,\omega_{dp})}{1+\mathcal{R}^2+2\mathcal{R}\cos\Delta\varphi(\omega_1,\omega_2,\omega_2)}\right]\times\tau_{\text{gen}}(\omega_1,\omega_2,\omega_{dp})+\left[\frac{\mathcal{R}^2+\mathcal{R}\cos\Delta\varphi(\omega_1,\omega_2,\omega_{dp})}{1+\mathcal{R}^2+2\mathcal{R}\cos\Delta\varphi(\omega_1,\omega_2,\omega_{dp})}\right]\times\tau_{\text{refl}}(\omega_1,\omega_2,\omega_{dp}). \qquad (71)$$

As expected, Eq. (71) reduces to $\tau_{dp}=\tau_{\text{gen}}$ when $\mathcal{R}\to 0$ and to $\tau_{dp}=\tau_{\text{refl}}$ when $\mathcal{R}\to\infty$. However, for intermediate values of $\mathcal{R}$, the phase derivative is in general neither a monotonically varying nor even a bounded function.

In order to study in detail the effects of fine structure on the phase derivative, the case of the fixed ratio will be considered. Under the assumption of scale invariance, $\tau_{\text{gen}}\approx 0$ and $\tau_{\text{refl}}\approx\tau_a\equiv-\varphi_a'(\omega)$. Since

$$\varphi_{dp0}(\omega_{dp},\omega_2)=2\int_0^{\hat{x}(\omega_2)}dx'\frac{k_0\omega_{dp}}{\sqrt{\omega_0(x')^2-\omega_{dp}^2}}\cong\frac{2k_0}{k_\omega}\sin^{-1}\frac{\omega_{dp}}{\omega_2}, \qquad (72)$$

where Eq. (18) has been used in obtaining this result, and $\omega_{dp}\ll\omega_{0c}$ has been assumed, it follows that

$$\Delta\varphi(\omega_{dp},\omega_2)\cong\varphi_a(\omega_{dp})+\text{constant}, \qquad (73)$$

for fixed $\omega_2/\omega_1$ (and hence, fixed $\omega_{dp}/\omega_2$). For simplicity, $\Delta\varphi\approx\varphi_a(\omega_{dp})$ will be assumed. In this case, Eq. (71) becomes

$$\tau_{\text{fixed-ratio}}(\omega_{dp})\cong\left[\frac{\mathcal{R}^2+\mathcal{R}\cos[(2\hat{k}/k_\omega)\log(\omega_{dp}/\omega_{0c})]}{1+\mathcal{R}^2+2\mathcal{R}\cos[(2\hat{k}/k_\omega)\log(\omega_{dp}/\omega_{0c})]}\right]\frac{2\hat{k}}{k_\omega\omega_{dp}}. \qquad (74)$$

This result is displayed in Fig. 1 for varying amounts of $\mathcal{R}$. Note that only for $\mathcal{R}\ll 1$ is the result expected for the case of a dominant generator-region component (Sec. IV) recovered. In addition, the phase derivative can vary quite wildly for $\mathcal{R}\approx 1$. Both of these observations also hold for the cases of the fixed-$f_1$ and fixed-$f_2$ paradigms. It was shown in Talmadge *et al.* (1999a) that the phase-derivative fine structure will correlate with the DPOAE level fine structure if $\mathcal{R}<1$ (that is, the maxima and minima will align), but will negatively correlate if $\mathcal{R}>1$ (the phase derivative maxima will correspond to level minima and vice versa). This pattern of positively and negatively correlating fine structure was also observed experimentally in Talmadge *et al.* (1999a).

The results obtained in this section were derived under the assumption that the denominator contribution to the fine structure in Eq. (6) could be ignored. The inclusion of this contribution makes the results correspondingly more complex, both for level and phase-derivative fine structures, as will be demonstrated in a paper now in preparation.

## VII. NUMERICAL TESTS AND VALIDATION OF THE APPROXIMATIONS

A significant number of approximations were made in order to obtain the quasianalytic results reported in Secs. III through VI. In this section, the effects of these assumptions will be explored in the context of a number of different numerical methods. In order to clarify the discussion, a short summary of the approximations used in obtaining the results is first given. The specific numerical methods used in the validation are then described. In the final subsection, numerical results which provide a test of the various assumptions made in the analyses are given. Alternative means of verifying some of the results are also given.

### A. Summary of the approximations

The approximations made in this analysis are listed in the order that they first appear in this paper.

(1) The cochlear model of Talmadge *et al.* (1998) is assumed.
(2) The validity of the results of Talmadge *et al.* (1998) given by Eqs. (7) and (8) is assumed. It should be noted that these results are themselves approximate, and depend on the following assumptions:
  (a) higher-order contributions in the perturbative expansion of the wave equation may be ignored;
  (b) the effects of nonlinearity on the activity patterns of the various frequency components may be neglected;
  (c) nonlinear cochlear wave reflection may be ignored.
(3) The cochlear basis functions are accurately described by the WKB approximation.
(4) Wave number scale invariance is assumed. This is equivalent to assuming that $\varepsilon_\gamma=\gamma_0(x)/\omega_0(x)$ is constant. This assumption results in the approximate relationships given separately in Eqs. (20) and (36).
(5) The neglect of the effects of time-delayed stiffness on DPOAE phase.
(6) The neglect of the influence of the middle ear on the DPOAE phase. This amounts to approximating the phase of $P_l(\omega_1,\omega_2,\omega_{dp})$ by the phase of $I_r(\omega_1,\omega_2,\omega_{dp})$.
(7) The neglect of the effects of damping on DPOAE phase in order to obtain a closed-form result for $\tau_{\text{gen}}(\omega_1,\omega_2,\omega_{dp})$ [see Eq. (50)].

(8) The neglect of the effects of the base (finite $\omega_{0c}$). This approximation plus Eqs. (39) and (40) give Eqs. (52) and (53), respectively.

## B. Numerical methods

The effects of these approximations will be systematically investigated by comparing the quasianalytic result with numerical ones based on the underlying cochlear model. In particular, the differences among the models will be explored by separately comparing the values of $\tau_{\text{fixed-}f_1}$ and $\tau_{\text{fixed-}f_2}$, as well as $\mathcal{S}(\omega_{dp})$ given by Eq. (55). In addition, for the study of the possible effects of the ''filter build-up time,'' alternative numerical tests are reported. These will be described separately in Sec. C.

Four different numerical computations for $\tau_{\text{fixed-}f_1}$, $\tau_{\text{fixed-}f_2}$, and $\mathcal{S}$ versus frequency were carried out. In all calculations, the primary frequency ratio was $f_2/f_1=1.22$, and derivatives were evaluated numerically using the centered-difference method. For the computation of $\mathcal{S}$, the results of these separate numerical calculations were compared to the predictions of DPOAE phase scale invariance ($\mathcal{S}\equiv 1$) and wave number scale invariance [Eq. (56)]. The four computations are described below.

(I) In order to make as few assumptions as possible, time-domain integration for a limited number of discretization points was employed. A cochlear model with 4096 sections, and a time step of $2.5\times 10^{-6}$ s was used. The model is that described in Talmadge *et al.* (1998) and includes time-delayed stiffness of the form described by Zweig (1991). However, for this analysis, the basilar membrane stiffness function was smoothly varying, instead of having embedded roughness as was the case in Talmadge *et al.* (1998). This simplification prevented any significant reflection-site DPOAE component from being present in the ear canal. The total integration time was 150 ms, and the final 75 ms of the simulated ear-canal signal were analyzed for the DPOAE signal. The effective stimulation levels for this model were $L_1=35$ dB SPLm and $L_2=35$ dB SPLm. The principal advantages of this approach are that it does not rely on perturbation theory and that it is able to account for distortion of the basilar membrane wave functions as a function of stimulus level. It can also account for secondary distortion product generation via the interaction of other DP products with each other and with the primaries. Its chief disadvantages are (i) that its accuracy decreases with increasing frequency of the primaries, as a result of the fixed step size of the time integration, (ii) the large numerical error for low-frequency components due to the finite-time interval for integration and the nonexact convergence of the numerical results to the steady-state solutions, and (iii) the large amount of required computing time. Nonetheless, it is expected that the utility of this approach should compare well with that of other numerical approaches for the midrange of frequencies (approximately 1000–5000 Hz).

(II) As an intermediate test, the frequency-domain perturbative DP wave equation (see the Appendix) was numerically solved, using 40 960 sections. The model was again that of Talmadge *et al.* (1998) except that it had a smooth basilar membrane stiffness function. The chief advantage of this procedure is its speed, which allows numerical integration with a finer grid spacing to be performed, and the phase derivatives to be obtained for a much larger number of DPOAE frequencies. The main disadvantage is that it cannot take into account nonlinear distortion of the basilar membrane traveling waves, so that results based on this approach must be regarded as the low stimulation level limit of those obtained from the time-domain integration approach. This method is equivalent to using approximations 1 and 2 above.

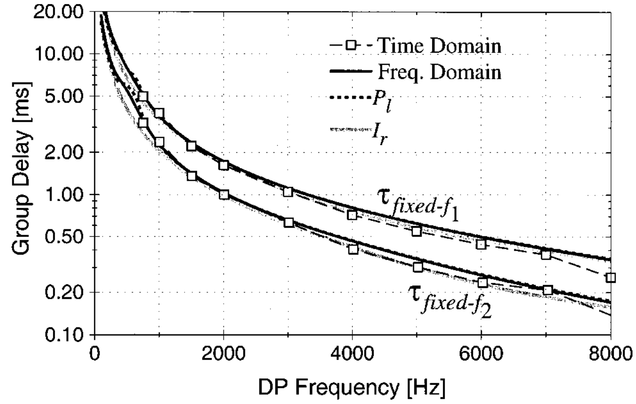


FIG. 2. Plots of $\tau_{\text{fixed-}f_1}$ and $\tau_{\text{fixed-}f_2}$ versus DP frequency for various numerical solutions. The time-domain solution corresponds to primary levels $L_1=L_2=15$ dB SPL. The ''frequency domain'' solution corresponds to the perturbative solution discussed in the text. The solution $P_l$ corresponds to the direct computation of Eq. (7) using the WKB approximation, and $I_r$ corresponds to the direct integration of Eq. (22). As discussed in the text, the use of just $I_r$ to compute the DPOAE phase corresponds to the neglect of middle- and outer-ear effects.

(III) The integral $F_r$ of Eq. (10) was numerically integrated using the basis functions $\psi_{r,l}(x,\omega)$ given by the WKB approximation [Eqs. (12) and (13)]. Because this model is just the Green's function formulation of the frequency-domain perturbative DP wave equation, this procedure can be considered to be nearly identical to that of method II except for the use of the WKB solutions in obtaining the numerical results. This method is equivalent to using approximations 1–3 above.

(IV) The integral $F_r$ was approximated by $I_r$ using Eq. (20). The integral $I_r$ [Eq. (22)] was numerically integrated, using the basis functions $\psi_{r,l}(x,\omega)$ given by the WKB approximation [Eqs. (12) and (13)]. Moreover, $I_r$ does not include effects of the middle ear, so that it cannot be used for frequencies near the middle-ear resonance frequency (500–1000 Hz). It should be noted that for this reason, the quasianalytic results reported upon in this paper are also not valid in this frequency range. This method is equivalent to using approximations 1–6 above.

## C. Results

A large number of numerical tests were run during the validation process for this manuscript. As a consequence, a representative but relatively small fraction of the various numerical results will be reported here.

The results of comparisons of the values of $\tau_{\text{fixed-}f_1}$ and $\tau_{\text{fixed-}f_2}$ based on methods I–IV are shown in Fig. 2. These

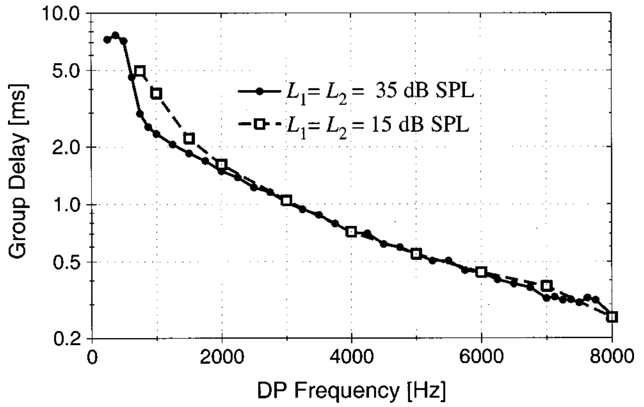


FIG. 3. Effect of level on $\tau_{\text{fixed-}f_1}$. Shown are the computed values based upon the time-domain solutions for primary levels $L_1 = L_2 = 15$ dB SPL and $L_1 = L_2 = 35$ dB SPL. As would be expected from the influence of the middle ear, the dominant effect of level is (initially) seen for DPOAE frequencies in the range 500–1500 Hz.

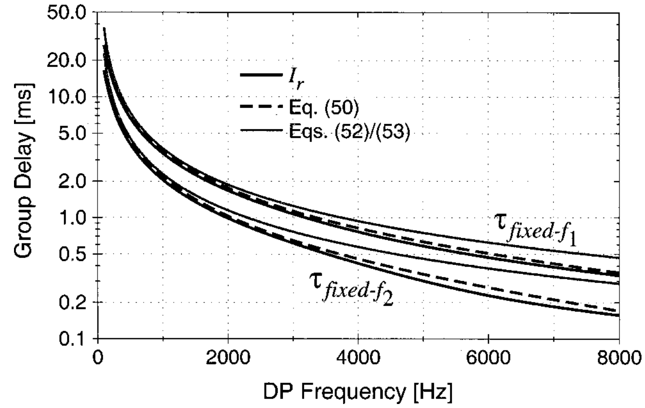


FIG. 4. Comparison of $\tau_{\text{fixed-}f_1}$ and $\tau_{\text{fixed-}f_2}$ computed from $I_r$ to those computed using Eq. (50) and using Eqs. (52) and (52).

results demonstrate the approximate equivalence of the predictions of the mathematical model of Talmadge *et al.* (1998) to those of its approximated forms given by approximations 1–6. It should be noted that the cochlear model used in all four calculations included scale-invariance violating terms for both the place-frequency map and the damping function, as described in Talmadge *et al.* (1998). Consequently the fact that, unlike the other approaches, the direct integration of $I_r$ did not yield significant symmetry violation in the frequency range 500–1000 Hz suggests that the middle and outer ear have a larger effect on the DPOAE phase than the scale-invariant violations of the cochlea at low frequencies.

One of the chief limitations of the approximate forms (frequency-domain perturbative calculation, WKB approximation of $P_l$, etc.) is the neglect of the frequency shift of the activity pattern maximum of the $f_2$ primary. The middle ear will transmit acoustic signals more efficiently near its characteristic frequency. Thus, for a fixed ear-canal signal level, there will be a larger basilar membrane amplitude of motion when the signal is near the middle-ear characteristic frequency than when it is well away from that characteristic frequency. If the basilar membrane nonlinear saturating level ($b_{nl}$) is approximately independent of frequency, as it is in the cochlear models used in this analysis, then it is expected that the effect of the shift of the activity pattern maximum on the phase derivatives will be largest for frequencies near the middle-ear characteristic frequency. This observation is borne out in Fig. 3, where $\tau_{\text{fixed-}f_1}$ is compared for $L_1 = L_2 = 15$ dB SPL to $L_1 = L_2 = 35$ dB SPL. In this figure, the two computed values of $\tau_{\text{fixed-}f_1}$ are seen to diverge only in the range of 500–1500 Hz.

In Fig. 4, the values of $\tau_{\text{fixed-}f_1}$ and $\tau_{\text{fixed-}f_2}$ calculated from $I_r$ are compared to those computed using Eq. (50) and using Eqs. (52) and (53). As discussed above, Eqs. (52) and (53) give the results for a DPOAE wave number scale-invariant model in which the contribution to the phase from the base of the cochlea is neglected, whereas Eq. (50) also corresponds to a wave number scale-invariant model which includes the effect of the cochlear base. In both cases, damping contributions to the DPOAE phase have been neglected. As expected, at high frequencies the results based on Eqs. (52) and (53) deviate substantially from those based on Eq. (50) and $I_r$. However, the agreement between the phase derivatives computed using $I_r$ and Eq. (50) is quite reasonable, given the level of approximation used to obtain Eq. (50).

Figure 5 displays the computed value of $\mathcal{S}$ from all four numerical approaches. Above approximately 6500 Hz, the time-domain integration error becomes evident for method 1. However, for midrange values (~750–4500 Hz in this simulation), the four approaches give results within approximately 10% with respect to the value of $\mathcal{S}$, and confirm the validity of the approximations used in this analysis for these frequencies. It should be noted that the deviations of $\mathcal{S}$ from 1 are not inconsistent with the model of Talmadge *et al.* (1998), but rather with the more restrictive assumption of DPOAE phase scale invariance. Also, differences among the four approaches are expected because of the varying levels of approximations used in obtaining the results.

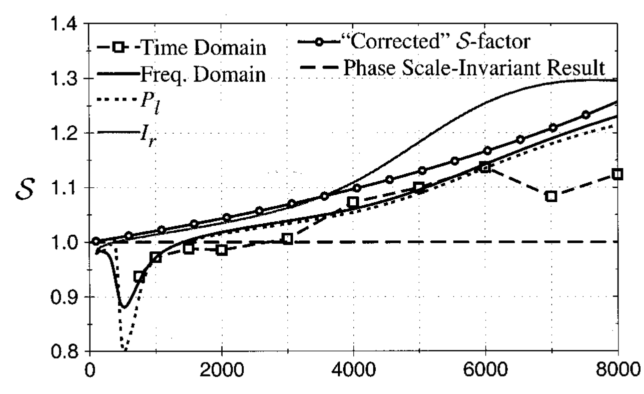


FIG. 5. Plot of $\mathcal{S}$ versus DPOAE frequency, for various numerical solutions. The time-domain, frequency-domain, $P_l$, and $I_r$ solutions are the same as discussed in the caption for Fig. 2. The factor $\mathcal{S}=1$ corresponds to a phase scale-invariant model, and the "corrected" $\mathcal{S}$ factor corresponds to the computed $\mathcal{S}$ factor corrected for the effect of the cochlear base ($\omega_{0c} < \infty$). The large deviations of the computed values of $\mathcal{S}$ from unity for frequencies near 500 Hz are a result of the scale-invariance violations caused by the middle ear.

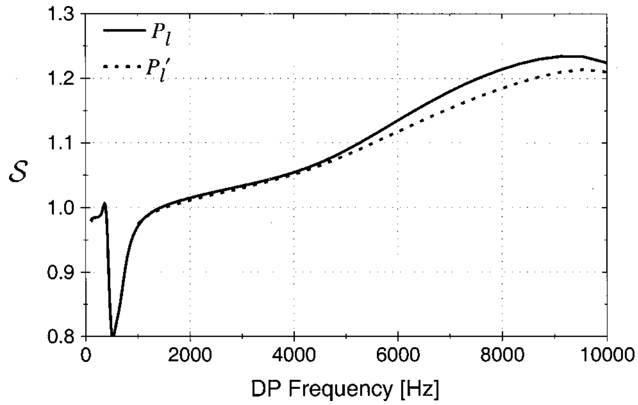


FIG. 6. Comparison of the predicted values of $\mathcal{S}$ versus DPOAE frequency for the perturbative frequency-domain solution $P_l$ and for the modified solution $P_l'$, which is $P_l$ evaluated using $|\Delta(x,\omega)|$ as discussed in the text.

Further insight into the nature of the dependence of the phase derivatives on the basilar membrane damping $\gamma_0(x)$ (and hence on the ''filter build-up'') is obtained by comparing the results of the perturbative frequency-domain calculation, $P_l$, with $P_l'$, which is the integration of $P_l$ carried out with $\Delta^*_{sm}(x,\omega_2)$ replaced by its magnitude. This was done so as to assess the importance of the resonant phase behavior of $\Delta^*_{sm}(x',\omega_2)$ on the phase derivatives of $P_l$, which was addressed in an approximate analytic approach in Sec. III. For this evaluation, the integral is labeled as $P_l'$ in Fig. 6. The fact that only a small effect on $\mathcal{S}$ is observed implies that the filter build-up time cannot be the principal explanation for the observed differences between $\tau_{\text{fixed-}f_1}$ and $\tau_{\text{fixed-}f_2}$.

## VIII. THE RELATIONSHIP BETWEEN THE FIXED-$f_1$ AND FIXED-$f_2$ PHASE DERIVATIVES FOR THE HIGHER-ORDER APICAL DPOAES

[The results obtained in this section have been previously considered in the context of scale invariance by Schneider *et al.* (1999b). See also Prijs *et al.* (2000).] Under the assumption that the generation region component of the DPOAE is dominant, a relationship between $\tau_{\text{fixed-}f_1}$ and $\tau_{\text{fixed-}f_2}$ can be obtained by assuming that the phase of the apical DPOAE from the generator site is scale invariant. In the case of an apical DPOAE of frequency $\omega_{dp}=n\omega_1-(n-1)\omega_2$, scale invariance implies,

$$\varphi_{\text{gen}}(\omega_1,\omega_2,\omega_{dp})=\varphi_{\text{gen}}\left(\frac{\omega_1}{\omega_2},\frac{\omega_{dp}}{\omega_2}\right). \tag{75}$$

Equation (75) is sufficient to establish a general relationship between the two phase derivatives for arbitrary apical order and for an arbitrary form of the nonlinearity, except that the nonlinearity must be of a form that can produce cubic distortion products. With the introduction of the scaling variables

$$\beta_1\equiv\frac{\omega_1}{\omega_2},\quad \beta_{dp}\equiv\frac{\omega_{dp}}{\omega_2}, \tag{76}$$

Eq. (75) becomes

$$\omega_{gen}(\omega_1,\omega_2,\omega_{dp})=\varphi_{\text{gen}}(\beta_1,\beta_{dp}). \tag{77}$$

Evaluating the partial derivatives of $\varphi_{\text{gen}}$ with respect to $\omega_1$ and $\omega_2$ gives

$$\left(\frac{\partial\varphi_{\text{gen}}}{\partial\omega_1}\right)_{\omega_2}=\frac{1}{\omega_2}\frac{\partial\varphi_{\text{gen}}}{\partial\beta_1}+\frac{n}{\omega_2}\frac{\partial\varphi_{\text{gen}}}{\partial\beta_{dp}},$$
$$=\frac{1}{\omega_2}\left[\frac{\partial\varphi_{\text{gen}}}{\partial\beta_1}+n\frac{\partial\varphi_{\text{gen}}}{\partial\beta_{dp}}\right], \tag{78}$$

and

$$\left(\frac{\partial\varphi_{\text{gen}}}{\partial\omega_2}\right)_{\omega_1}=-\frac{\omega_1}{\omega_2^2}\frac{\partial\varphi_{\text{gen}}}{\partial\beta_1}-\left[\frac{n-1}{\omega_2}+\frac{\omega_{dp}}{\omega_2^2}\right]\frac{\partial\varphi_{\text{gen}}}{\partial\beta_{dp}},$$
$$=-\frac{\omega_1}{\omega_2^2}\frac{\partial\varphi_{\text{gen}}}{\partial\beta_1}-\left[\frac{(n-1)\omega_2}{\omega_2^2}+\frac{n\omega_1-(n-1)\omega_2}{\omega_2^2}\right]\frac{\partial\varphi_{\text{gen}}}{\partial\beta_{dp}},$$
$$=-\frac{\omega_1}{\omega_2^2}\left[\frac{\partial\varphi_{\text{gen}}}{\partial\beta_1}+n\frac{\partial\varphi_{\text{gen}}}{\partial\beta_{dp}}\right]. \tag{79}$$

Next, evaluating the phase derivatives with respect to $\omega_{dp}=n\omega_1-(n-1)\omega_2$ gives

$$\tau_{\text{fixed-}f_1}(\omega_{dp})=-\left(\frac{\partial\varphi_{\text{gen}}}{\partial\omega_{dp}}\right)_{\omega_1}$$
$$=\frac{1}{n-1}\left(\frac{\partial\varphi_{\text{gen}}}{\partial\omega_2}\right)_{\omega_1}$$
$$=-\frac{\omega_1}{(n-1)\omega_2^2}\left[\frac{\partial\varphi_{\text{gen}}}{\partial\beta_1}+n\frac{\partial\varphi_{\text{gen}}}{\partial\beta_{dp}}\right], \tag{80}$$

$$\tau_{\text{fixed-}f_2}(\omega_{dp})=-\left(\frac{\partial\varphi_{\text{gen}}}{\partial\omega_{dp}}\right)_{\omega_2}$$
$$=-\frac{1}{n}\left(\frac{\partial\varphi_{\text{gen}}}{\partial\omega_1}\right)_{\omega_2}$$
$$=-\frac{1}{n\omega_2}\left[\frac{\partial\varphi_{\text{gen}}}{\partial\beta_1}+n\frac{\partial\varphi_{\text{gen}}}{\partial\beta_{dp}}\right]. \tag{81}$$

Combining Eqs. (80) and (81) gives

$$\frac{\tau_{\text{fixed-}f_1}(\omega_{dp})}{\tau_{\text{fixed-}f_2}(\omega_{dp})}=\frac{n}{n-1}\frac{\omega_1}{\omega_2}. \tag{82}$$

Finally, it should be noted that if the phase derivatives are evaluated in terms of $\omega_1$ and $\omega_2$ directly, as has been typically done in previous studies (e.g., O'Mahoney and Kemp, 1995; Bowman *et al.*, 1997, 1998), then the ratio of these phase derivatives is just

$$\frac{(\partial\varphi_{\text{gen}}/\partial\omega_2)_{\omega_1}}{(\partial\varphi_{\text{gen}}/\partial\omega_1)_{\omega_2}}\cong\frac{\omega_1}{\omega_2}, \tag{83}$$

independent of the DPOAE order.

## IX. SUMMARY AND CONCLUSIONS

The importance of considering the variation of both the DPOAE phase and amplitude with frequency when studying DPOAE fine structure effects has been previously demonstrated by Talmadge *et al.* (1999a). Their results, together with studies such as those of O'Mahoney and Kemp (1995); Bowman *et al.* (1997,1998), Schneider *et al.* (1999a), and Kemp and Knight (2000), underscore the importance of understanding in more detail the proper interpretation of the DPOAE phase and its derivatives. Recent theoretical studies (Talmadge *et al.*, 1999b; Tubis *et al.*, 2000) have already demonstrated that great care must be exercised when attempting to relate DPOAE phase derivatives to the relevant latencies.

In this paper, expressions describing the phase derivatives $\tau_{\text{fixed-ratio}}$, $\tau_{\text{fixed-}f_1}$, and $\tau_{\text{fixed-}f_2}$ have been obtained in Secs. II and III using the theoretical framework of Talmadge *et al.* (1998). In Sec. III, an approximate analytical argument was used to show that the contributions to these phase derivatives from the resonant cochlear partition phase behavior in the generation region (around the $f_2$ tonotopic site) are small. This contribution is presumably what Bowman *et al.* (1997,1998) associate with the filter build-up time. In Sec. VII, this conclusion was reinforced using four different numerical approaches. It was shown in particular that the cochlear filter contributions to $\mathcal{S}$, and hence to the ratio of $\tau_{\text{fixed-}f_1}$ to $\tau_{\text{fixed-}f_2}$, are small. On this basis, it was concluded that these contributions could not be the main source for the substantial differences between these phase derivatives that have been found experimentally.

In Sec. IV, it was shown that wave number scale invariance applied to the generator region DPOAE component is sufficient to give a general result [Eq. (54)] for the ratio of $\tau_{\text{fixed-}f_1}$ to $\tau_{\text{fixed-}f_2}$, which is in fairly good agreement with experimental data for $f_2/f_1$ in the range of 1.1 to 1.3. This result has been previously derived by Schneider *et al.* (1999b). Because of the restrictive nature of the assumptions made in obtaining this relationship, however, the effect of small scale-invariance violating contributions cannot be modeled directly using this approach. This situation is in contrast to the results on scale-invariance breaking that can be obtained from the auditory periphery model discussed in Sec. VII.

In the case of a dominant reflection region DPOAE component, it was shown in Sec. V that for any of the measurement paradigms (fixed-$f_1$, fixed-$f_2$, fixed-$f_2/f_1$), the phase derivative is approximately the round-trip travel time for a cochlear wave of the DPOAE frequency between the cochlear base and the DP tonotopic site. When the phase derivative of the DPOAE resulting from the combined effects of the generator and reflection region components was considered in Sec. VI, it was found that significant variations in the DPOAE phase and amplitude can occur as the DPOAE frequency is varied. These variations constitute the DPOAE fine structure, which has been studied in a limited theoretical context and experimentally corroborated by Talmadge *et al.* (1999a).

The numerical results presented in Sec. VII support the approximate validity of scale invariance over a wide range of frequencies (roughly 1000–5000 Hz for humans) for a cochlear model that manifestly contains scale-invariance violations. Some of the variations of the numerical results from the simple analytical expressions obtained in Sec. IV using the assumption of scale invariance were also shown to be explainable by the inclusion of a small scale-invariance violating term related to the cochlear base. The numerical results also suggest that the assumption of scale invariance can be used to describe qualitative properties of cochlear models, even when these models exhibit significant scale-invariance violations, such as a non exponential place-frequency map or the inclusion of a realistic middle ear. An important limitation of the present paper is the restriction of the analysis for the DPOAE phase-derivative fine structure to the case in which the effects of multiple internal cochlear wave reflections can be ignored [that is, it was assumed that $|R_a R_b| \ll 1$ in Eq. (6)]. Consideration of the effects of fine structure on the phase derivative and level of $P_{ss}^{dp}$ will be deferred to a future paper.


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## APPENDIX: PERTURBATIVE SOLUTION FOR DPOAES

In this section, the perturbative solution for the $2\omega_1 - \omega_2$ DP will be illustrated. In the one-dimensional cochlear model, the steady-state transpartition pressure wave $P_d(x,\omega)$ of frequency $\omega$ is described by the wave equation

$$P_d''(x,\omega) + k^2(x,\omega) P_d(x,\omega) = 0, \tag{A1}$$

where $k(x,\omega)$ is the complex wave number of the traveling wave given by Eq. (14). The $\Delta(x,\omega)$ function that appears in this latter equation is defined by

$$\Delta(x,\omega) = \frac{1}{\sigma_{bm}} \frac{P_d(x,\omega)}{\xi(x,\omega)}. \tag{A2}$$

In order to generalize this expression to the case for an internally generated DP, it is necessary to use the time-domain formulation of the pressure wave equation, namely (e.g., Talmadge *et al.*, 1998)

$$\frac{\partial^2 P_d(x,t)}{\partial x^2} = -k_0^2 \sigma_{bm} \ddot{\xi}(x,t), \tag{A3}$$

$$\ddot{\xi}(x,t) + \gamma_{bm}(x,\xi,\dot{\xi})\,\dot{\xi}(x,t) + \omega_{bm}^2(x,\xi,\dot{\xi})\,\xi(x,t) = \frac{1}{\sigma_{bm}} P_d(x,t), \tag{A4}$$

where $\dot{A}(t)$ refers to differentiation with respect to time, and where $\gamma_{bm}$ and $\omega_{bm}$ are, respectively, the basilar membrane damping and stiffness functions. The steady-state solution for the $\omega_{dp} = 2\omega_1 - \omega_2$ DP can then be found by writing

$$\xi(x,t) \approx \xi_1(x,\omega_1)e^{i\omega_1 t} + \xi_2(x,\omega_2)e^{i\omega_2 t} + \xi_{dp}(x,\omega_{dp})e^{i\omega_{dp} t} + \text{complex conjugate}, \quad (A5)$$

$$P_d(x,t) \approx P_1(x,\omega_1)e^{i\omega_1 t} + P_2(x,\omega_2)e^{i\omega_2 t} + P_{dp}(x,\omega_{dp})e^{i\omega_{dp} t} + \text{complex conjugate}, \quad (A6)$$

where $\xi_{1,2,dp}$ and $P_{1,2,dp}$ are, respectively, the $\omega_1$, $\omega_2$, and $\omega_{dp}$ Fourier components of $\xi$ and $P_d$.

As in the previous sections, a simple quadratic nonlinear (''Van der Pol-type'') damping is assumed, and possible stiffness feedback terms are neglected. Then, Eq. (A4) becomes

$$\ddot{\xi}(x,t) + \gamma_0(x)\left[1 + \frac{\xi^2(x,t)}{b_{nl}^2}\right]\dot{\xi}(x,t) + \omega_0^2(x,\xi,\dot{\xi})\xi(x,t) = \frac{1}{\sigma_{bm}} P_d(x,t), \quad (A7)$$

where $\gamma_0(x)$ and $\omega_0^2(x)$ are the usual passive linear damping and stiffness, respectively, of the basilar membrane, and $b_{nl}$ is the nonlinear saturation level.

Equations (A3) and (A7) can be solved perturbatively if it is assumed that the nonlinearity is weak; that is, $\xi^2(x,t)/b_{nl}^2 \ll 1$. It will be assumed that this nonlinear contribution to Eq. (A7) as well as $\xi_{dp}(x,\omega)$ and $P_{dp}(x,\omega)$ are of first order in perturbation theory. Using the perturbation parameter $\varepsilon_p$ to track the order of perturbation, then

$$\xi_1(x,\omega) = \sum_{n=0}^{\infty} \varepsilon_p^n \xi_1^{(n)}(x,\omega), \qquad P_1(x,\omega) = \sum_{n=0}^{\infty} \varepsilon_p^n P_1^{(n)}(x,\omega), \quad (A8)$$

$$\xi_2(x,\omega) = \sum_{n=0}^{\infty} \varepsilon_p^n \xi_2^{(n)}(x,\omega), \qquad P_2(x,\omega) = \sum_{n=0}^{\infty} \varepsilon_p^n P_2^{(n)}(x,\omega), \quad (A9)$$

$$\xi_{dp}(x,\omega) = \sum_{n=1}^{\infty} \varepsilon_p^n \xi_{dp}^{(n)}(x,\omega), \qquad P_{dp}(x,\omega) = \sum_{n=1}^{\infty} \varepsilon_p^n P_{dp}^{(n)}(x,\omega), \quad (A10)$$

where the notation $F^{(n)}(x,\omega)$ is used to denote the *n*th order of the perturbation theory. Also, by assumption of a weak nonlinearity, Eq. (A10) contains no $n=0$ perturbative contribution. Equation (A7) also becomes

$$\ddot{\xi}(x,t) + \gamma_0(x)\dot{\xi}(x,t)\omega_0^2(x,\xi,\dot{\xi})\xi(x,t) = \frac{1}{\sigma_{bm}} P_d(x,t) - \varepsilon_p \frac{\xi^2(x,t)}{b_{nl}^2}\dot{\xi}(x,t), \quad (A11)$$

where the extra factor of $\varepsilon_p$ has been inserted to denote the assumed first-order perturbative term. Insertion of Eqs. (A8)–(A10) into Eqs. (A3) and (A11) gives the lowest-order pressure wave equations

$$\frac{\partial P_1^{(0)}(x,\omega_1)}{\partial x^2} + k^2(x,\omega_1)P_1^{(0)}(x,\omega_1) = 0, \quad (A12)$$

$$\frac{\partial P_2^{(0)}(x,\omega_2)}{\partial x^2} + k^2(x,\omega_2)P_2^{(0)}(x,\omega_2) = 0, \quad (A13)$$

$$\frac{\partial P_{dp}^{(0)}(x,\omega_{dp})}{\partial x^2} + k^2(x,\omega_{dp})P_{dp}^{(0)}(x,\omega_{dp}) = \rho_{dp}(x,\omega_1,\omega_2,\omega_{dp}), \quad (A14)$$

$$\rho_{dp}(x,\omega_1,\omega_2,\omega_{dp}) = \frac{i\sigma_{bm}\gamma_0(x)k_0^2\omega_{dp}^3[\xi_1^{(0)}(x,\omega_1)]^2[\xi_2^{(0)}(x,\omega_2)]^*}{\Delta(x,\omega_{dp})b_{nl}^2}. \quad (A15)$$

Equations (A12)–(A15) have been solved in Talmadge *et al.* (1998) with $\gamma_0(x) = \varepsilon_\gamma \omega_0(x)$ using the basis function formalism, and it is this solution that is given in Sec. II of this paper.


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